HEATING OF A MEDIUM AS A RESULT OF JOULE ENERGY DISSIPATION

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 28-33, 1965

It is well known that when a magnetic field is present and electric currents flow through a gas, terms over and above those present in the case of ordinary thermal conductivity appear in the heat flux density vector. If the gas is dense enough and the magnetic field not over large, then the anisotropy caused by the magnetic field may be neglected. However, for a sufficiently large electric current a term proportional to the temperature and to the current density vector remains in the heat flux density vector. This effect explains, for example, the asymmetry of heat fluxes in the electrodes of a continuously operated electromagnetic accelerator (1).

We shall consider this situation in relation to the example of a fully ionized quasineutral gas with identical electron and ion temperatures.

Let the density of the electric current density j, flowing through the gas, be of the order 10 amp/cm^2 and the external magnetic field be equal to zero. We find from Maxwell's equations that the magnetic field strength H is of the order $4\pi c^{-1} j l$, where l is the characteristic dimension of field variation, c is the velocity of light in a vacuum. Taking $l \sim 10$ cm, we find

$$\omega_e \tau_e \approx \frac{0.77}{\lambda / 10} \frac{10^{14} T_э^{3/2}}{n_e} \qquad \left(\tau_e = \frac{3.5 \cdot 10^4 T_э^{3/2}}{n_e \lambda / 10}, \quad \omega_e = \frac{eH}{m_e c}\right).$$

Here λ in the Coulomb logarithm, T_2 is the temperature in eV, e and m_e are the electronic charge and mass, respectively, n_e is the electron density, τ_e is the scattering time for electrons on ions. Since $(0.1\lambda) \sim 1$, for $T_э \sim 1$ and $n_e \gtrsim 10^{16}$ the parameter $\omega_e \tau_e \ll 1$. This enables us to write the heat flux density vector in the form [2]

$$\mathbf{q} = -3.21 e^{-1} KT \mathbf{j} - 3.16 K^2 T \tau_e n_e m_e^{-1} \nabla T .$$

Here t is the temperature in degrees, and K is Boltzmann's constant. For j ~ 10 amp/cm^2 the ratio $2a$ of the first term to the second is of the order

$$2a \sim 1.2\,(\lambda / 10)\,(10^4 / |\nabla T|)\, T_э^{-3/2}.$$

In this estimate ∇T has dimensions degree/cm. For $T_э \sim 1$ and $|\nabla T| \sim 10^4$ the parameter $2a$ is of order unity. At larger flux densities the magnitude of this parameter increases.

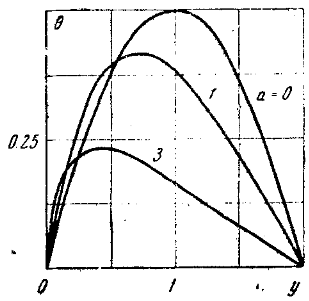

Fig. 1

In order to study the effects caused by the presence of the term proportional to j, in the heat flux density vector, we consider the problem of heating of the medium as a result of a current flow in which the vector **q** has the form

$$\mathbf{q} = -k \nabla T - b \mathbf{j} T \qquad (b = \text{const}) . \tag{1}$$

where k is the thermal conductivity. We discuss the possibility of using the solution obtained to estimate the size of the frictional force and the intensity of heat transfer for a medium flowing in an electromagnetic plasma accelerator.

We shall consider a flat channel $0 < Y < 2h$ with wall-electrodes, filled with a motionless conducting medium having a temperature T_j = const at the moment t = 0. Let the medium be heated as a result of the flow of electric current in the direction Y with constant current density. We shall designate by w = w (t) the Joule dissipation per unit volume in unit time. We shall consider that the channel walls are maintained at a constant temperature T_w in the process of heating. For constant k, density of medium ρ and heat capacity c_p, the heat conduction equation in dimensionless quantities assumes the form

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial y^2} + 2a \frac{\partial \vartheta}{\partial y} + v, \quad \vartheta(0, y) = \vartheta_i \qquad \text{for } 0 < y < 2$$
$$\vartheta(\tau, 0) = 0, \quad \vartheta(\tau, 2) = 0 \tag{2}$$
$$\left(\vartheta = \frac{T - T_w}{T_w}, \quad Y = hy, \quad t = \frac{h^2 \rho c_p \tau}{k}, \quad 2a = \frac{bjh}{k}, \quad v = \frac{wh^2}{kT_w}\right)$$

in the case when the vector **q** is expressed by Eq. (1).

The heat fluxes G_+ and G_- conducted through unit surface of the lower and upper electrodes in time t are equal to

$$G_+ = -\int_0^t q_y(t, 0)\, dt = \rho c_p h T_w g_+,$$
$$G_- = \int_0^t q_y(t, 2h)\, dt = \rho c_p h T_w g_-, \tag{3}$$
$$g_+ = 2a\tau + \int_0^\tau \left(\frac{\partial \vartheta}{\partial y}\right)_{y=0} d\tau, \qquad g_- = -2a\tau - \int_0^\tau \left(\frac{\partial \vartheta}{\partial y}\right)_{y=2} d\tau .$$

Here q_y is the component of the vector **q** on the Y axis.

Integrating equation (2) with respect to y from y = 0 to y = 2 and with respect to time, we obtain the integrated heat balance equation

$$2\vartheta_i + 2\int_0^\tau v\, d\tau = g_+ + g_- + \int_0^2 \vartheta\, dy. \tag{4}$$

If v → const for $\tau \to \infty$, then for $\tau \to \infty$ a stationary configuration is established, in which the temperature distribution has the form

$$\theta(y) = \frac{\vartheta(\infty, y)}{v} = \frac{1}{a}\left[\frac{1-\exp(-2ay)}{1-\exp(-4a)} - \frac{y}{2}\right],$$
$$L(a) = \frac{1}{v}\int_0^2 \vartheta(\infty, y)\,dy = \frac{2a - \operatorname{th} 2a}{2a^2 \operatorname{th} 2a}. \qquad (5)$$

Curves of the functions $\Theta(y)$ and $L(a)$ are given in Figs. 1 and 2. The solution of (5) shows that with the increase of parameter a the temperature level in the stationary state and the amount of heat which the medium acquires in the process of passing to the stationary state decrease. The temperature profiles are asymmetric with respect to the channel axis y = 1; the temperature maxima are displaced towards the anode side.

We shall seek a solution of equation (2) with the help of the Laplace transform

$$\vartheta^\circ(p, y) = \int_0^\infty \exp(-p\tau)\,\vartheta(\tau, y)\,d\tau.$$

When transformed, equation (2) and its solution have the form

$$\vartheta^{\circ\prime\prime} + 2a\vartheta^{\circ\prime} - p\vartheta^\circ = -Mp, \qquad \vartheta^\circ(0) = 0, \qquad \vartheta^\circ(2) = 0,$$
$$\vartheta^\circ = M(\operatorname{sh} 2\varkappa)^{-1}\{\operatorname{sh} 2\varkappa - e^{a(2-y)}\operatorname{sh}\varkappa y - e^{-ay}\operatorname{sh}[\varkappa(2-y)]\},$$
$$M = M(p) = p^{-1}(\vartheta_i + v_c V^\circ), \qquad \varkappa = \sqrt{p + a^2},$$
$$V^\circ = \int_0^\infty \exp(-p\tau)V(\tau)\,d\tau, \quad (v = v_c V, \qquad v_c = \text{const}). \ (6)$$

For the function $\varkappa(p)$ in the p plane with a cut along the negative axis from $p = -a^2$ to an infinitely distant point, a branch was chosen which is positive for real $p > -a^2$. Since $|\exp(-4\varkappa)| < 1$ for Re p > 0, the function $(\operatorname{sh} 2\varkappa)^{-1}$ may be expanded in the series

$$(\operatorname{sh} 2\varkappa)^{-1} = 2\exp(-2\varkappa)\sum_{k=0}^\infty \exp(-4k\varkappa) \qquad (\operatorname{Re} p > 0).$$

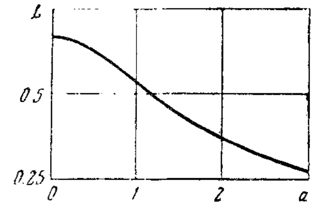

Fig. 2

By means of $r(t;\beta,\alpha)$ and $\mu(t;\beta,\alpha)$ we designate the inverses of the transforms

$$r^\circ = p^{-1}\exp[-\sqrt{\alpha(p+\beta)}],$$

$$\mu^\circ = p^{-1}V^\circ\exp[-\sqrt{\alpha(p+\beta)}] \quad (\alpha, \beta = \text{const}).$$

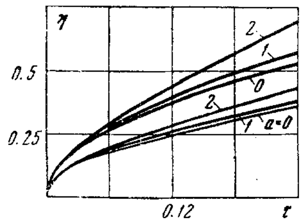

Fig. 3

Then replacing $(\operatorname{sh} 2\varkappa)^{-1}$ in expression (6) by the series indicated above, we may obtain the following expression for the function $\vartheta(\tau, y)$:

$$\vartheta(\tau, y) = \vartheta_i + \int_0^\tau v\,d\tau + \vartheta_i\Gamma[r] + v_c\Gamma[\mu],$$

$$\Gamma[\Lambda] =$$
$$= e^{-ay}\left\{\sum_{k=1}^\infty \Lambda[\tau; a^2, (4k-y)^2] - \sum_{k=0}^\infty \Lambda[\tau; a^2, (4k+y)^2]\right\} +$$
$$+ e^{a(2-y)}\left\{\sum_{k=0}^\infty \Lambda[\tau; a^2, (4k+2+y)^2] -\right.$$
$$\left. - \sum_{k=0}^\infty \Lambda[\tau; a^2, (4k+2-y)^2]\right\}. \qquad (7)$$

We find [3]

$$r(\tau; \beta, \alpha) = 0.5\,[\zeta(\tau; \beta, \alpha) + \delta(\tau; \beta, \alpha)],$$
$$\zeta = \exp(-\sqrt{\alpha\beta})\operatorname{Erf}(0.5\sqrt{\alpha/\tau} - \sqrt{\beta\tau}),$$
$$\delta = \exp\sqrt{\alpha\beta}\operatorname{Erf}(0.5\sqrt{\alpha/\tau} + \sqrt{\beta\tau})$$
$$\left(\operatorname{Erf} x = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)\,dx\right). \qquad (8)$$

If v = const, then

$$\mu(\tau; \beta, \alpha) = 0.5\tau(\zeta + \delta) - \frac{\sqrt{\alpha}}{4\sqrt{\beta}}(\zeta - \delta). \qquad (9)$$

In what follows all the equations will be given for the case v = const.

For $a = 0$ we have

$$\Gamma(\Lambda) = \sum_{k=1}^\infty (-1)^k \Lambda[\tau; 0, (2k-y)^2] -$$
$$- \sum_{k=0}^\infty (-1)^k \Lambda[\tau; 0, (2k+y)^2],$$
$$\mu(\tau; 0, \alpha) = (\tau + 0.5\alpha)\operatorname{Erf}(0.5\sqrt{\alpha/\tau}) -$$
$$- \pi^{-1/2}\sqrt{\alpha\tau}\exp(-\alpha/4\tau),$$
$$r(\tau; 0, \alpha) = \operatorname{Erf}(0.5\sqrt{\alpha/\tau}). \qquad (10)$$

The heat fluxes g_+ and g_-, determined from (3), (7)–(9), and the summed heat flux $Q = g_+ + g_-$ are equal to

$$g_+ = 2a\tau + N(\tau), \quad g_- = N(\tau) - 2a\tau - \vartheta_i A - vB,$$

$$N(\tau) = a\tau\vartheta_i + 0.5av\tau^2 + \vartheta_i[\psi_0 + 2E - 2F\exp 2a] +$$

$$+ v[\nu_0 + 2S - 2U\exp 2a], \quad E = \sum_{k=2,4,6,\ldots} \psi_k,$$

$$F = \sum_{k=1,3,5,\ldots} \psi_k, \quad S = \sum_{k=2,4,6,\ldots} \nu_k, \quad U = \sum_{k=1,3,5,\ldots} \nu_k,$$

$$\psi_k = \frac{\sqrt{\tau}}{\sqrt{\pi}} \exp\left(-\frac{k^2}{\tau} - a^2\tau\right) -$$

$$- 0.5k(\zeta + \delta) + \frac{1 + 2a^2\tau}{4a}(\zeta - \delta),$$

$$\nu_k = \frac{\sqrt{\tau}}{2\sqrt{\pi}}\left(\tau + \frac{1}{2a^2}\right)\exp\left(-\frac{k^2}{\tau} - a^2\tau\right) -$$

$$- 0.5k\left(\tau + \frac{1}{4a^2}\right)(\zeta + \delta) +$$

$$+ \frac{1}{8a}\left(2a^2\tau^2 + 2\tau + 2k^2 - \frac{1}{2a^2}\right)(\zeta - \delta),$$

$$\zeta = \zeta(\tau; a^2, 4k^2), \quad \delta = \delta(\tau; a^2, 4k^2),$$

$$A = 2a\tau - 4F\,\mathrm{sh}\,2a, \quad B = a\tau^2 - 4U\,\mathrm{sh}\,2a, \tag{11}$$

$$Q = 2\vartheta_i Q^* + 2vQ^{**}, \quad Q^* = \psi_0 + 2E - 2F\,\mathrm{ch}\,2a,$$

$$Q^{**} = \nu_0 + 2S - 2U\,\mathrm{ch}\,2a. \tag{12}$$

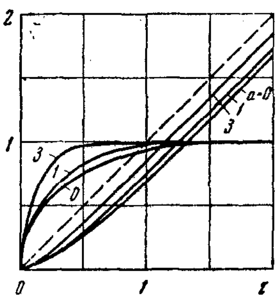

Fig. 4

For $a = 0$ we have

$$g_+ = g_- = g,$$

$$g = \vartheta_i\left(\psi_0 + 2\sum_{k=1}^{\infty}(-1)^k\psi_k\right) + v\left(\nu_0 + 2\sum_{k=1}^{\infty}(-1)^k\nu_k\right),$$

$$\psi_k = \psi_k^\circ = 2\sqrt{\tau/\pi}\exp(-k^2/\tau) - 2k\,\mathrm{Erf}(k/\sqrt{\tau}),$$

$$\nu_k = \nu_k^\circ = \frac{4\sqrt{\tau}}{3\sqrt{\pi}}\exp\left(-\frac{k^2}{\tau}\right)(\tau + k^2) -$$

$$- \mathrm{Erf}\left(\frac{k}{\sqrt{\tau}}\right)\left(2k\tau + \frac{4k^3}{3}\right). \tag{13}$$

If we seek the inverse transforms in the case where $a = 0$ with the help of the second expansion theorem [4], we obtain the following expression for the quantity g:

$$g = \vartheta_i - \frac{8\vartheta_i}{\pi^2}\sum_{\rho=0}^{\infty}\frac{\exp[-\frac{1}{4}\pi^2\tau(2\rho+1)^2]}{(2\rho+1)^2} + v\left(\tau - \frac{1}{3}\right) +$$

$$+ \frac{32v}{\pi^4}\sum_{\rho=0}^{\infty}\frac{\exp[-\frac{1}{4}\pi^2\tau(2\rho+1)^2]}{(2\rho+1)^4}. \tag{14}$$

It is convenient to use expression (13) for small τ, and expression (14) for large τ. For small a the functions ψ_k and ν_k may be represented in the following form.

$$\psi_k = \psi_k^\circ + a^2\left[\frac{2\sqrt{\tau}}{3\sqrt{\pi}}\exp\left(-\frac{k^2}{\tau}\right)(\tau + 4k^2) -\right.$$

$$\left. - 2k\,\mathrm{Erf}\left(\frac{k}{\sqrt{\tau}}\right)\left(\tau + \frac{4k^2}{3}\right)\right] + O(a^4),$$

$$\nu_k = \nu_k^\circ + a^2\{-2k\,\mathrm{Erf}(k/\sqrt{\tau})(\tfrac{1}{2}\tau^2 + \tfrac{4}{3}k^2\tau + \tfrac{2}{5}k^4) +$$

$$+ \exp(-k^2/\tau)(\tfrac{4}{5}k^4 + \tfrac{34}{15}k^2\tau + \tfrac{4}{15}\tau^2)\} + O(a^4). \tag{15}$$

For small τ, we have, neglecting terms of the order of $\exp(-1/\tau)$,

$$Q \approx 2\vartheta_i\psi_0 + 2v\nu_0 = 2\vartheta_i\left[\frac{\sqrt{\tau}}{\sqrt{\pi}}\exp(-a^2\tau) +\right.$$

$$\left. + \frac{1 + 2a^2\tau}{2a}\mathrm{erf}(a\sqrt{\tau})\right] + 2v\left[\frac{\sqrt{\tau}}{2\sqrt{\pi}}\left(\tau + \frac{1}{2a^2}\right)\exp(-a^2\tau) +\right.$$

$$\left. + \frac{1}{4a}\left(2a^2\tau^2 + 2\tau - \frac{1}{2a^2}\right)\mathrm{erf}(a\sqrt{\tau})\right]$$

$$(\mathrm{erf}\,x = 1 - \mathrm{Erf}\,x). \tag{16}$$

For small τ and a we find, using (15) and (16),

$$Q \approx \frac{4\vartheta_i\sqrt{\tau}}{\sqrt{\pi}}\left(1 + \frac{a^2\tau}{3}\right) + \frac{8v\tau\sqrt{\tau}}{3\sqrt{\pi}}\left(1 + \frac{a^2\tau}{5}\right). \tag{17}$$

The effectiveness of heat transfer to the walls is characterized by the quantity

$$\eta \approx \frac{Q}{2v\tau + 2\vartheta_i}, \tag{18}$$

which represents the ratio of thermal energy transferred through the walls in time τ to the sum of the energy arriving during this time and the thermal energy which the medium possessed when $\tau = 0$. The function $\eta(\tau)$ depends on two parameters: a and $s = \vartheta_i/v$. For $\tau \to \infty$ we have $(dQ/d\tau) \approx 2v$ and $\eta \to 1$. However, calculations show that $(dQ/d\tau) \approx 2v$ for $\tau > \tau_*$, where $\tau_* \sim 1$. The quantity τ_* is the time in which the stationary state is established. During this time the medium loses an amount of heat $2\vartheta_i$, and gains an amount $vL(a)$, equal to the difference between the heat which has arrived $2v\tau_*$ and the heat conducted through the walls $2vQ^{**}(\tau_*)$.

Results calculated from Eqs. (12)–(18) are shown in Figs. 3–5. The behavior of the quantity η is shown in Fig. 3 for small values of τ. For the three lower curves s = 0, while for the three upper s = 2. The effectiveness of heat transfer increases as the parameters a and s increase. The behavior of $Q^*(\tau)$ is represented in Fig. 4 by a series of curves having an asymptote at $\varphi(\tau) = 1$. The three curves, which become parallel to the bisectrix $\varphi(\tau) = \tau$ when $\tau \to \infty$, are graphs of the function $Q^{**}(\tau)$. For each curve of the first series we can indicate a point $\tau = \tau_*(\varepsilon)$ starting from which the difference between the ordinates of points on the curve and unity becomes less than $\varepsilon = o(1)$; for each curve of the second series we can indicate a point $\tau = \tau_*(\varepsilon)$ starting from which the difference between the angle of inclination of the curve to the τ axis and the angle 45° becomes less than ε. In accordance with Fig. 4 the time τ_* for transition to the stationary configuration decreases as a increases.

Figure 5 represents the behavior of $\eta(\tau)$ for $\vartheta_i = 0$, and various values of a. An increase in this parameter leads to an increase in the effectiveness of heat transfer.

It is useful to note that one may regard the thermal energy $2\Theta_i$ possessed by the gas when $\tau = 0$ as energy received from thermal sources having an intensity $v(\tau) = \vartheta_i\delta(\tau)$, where $\delta(\tau)$ is a delta-function. Thus the solution of (7)–(9) will also be a solution to the problem of the heating of the medium from a temperature $T_i = T_W$ as a result of the dissipation of energy with density

$$w = \frac{kT_w}{h^2}[v + \vartheta_i\delta(\tau)] \qquad (v = \text{const}).$$

In conclusion, we consider the possibility of using the solutions which have already been obtained to make a rough estimate of the magnitude of the frictional forces and heat transfer in electromagnetic accelerators having a constant cross section along their length. When a conducting gas is accelerated by a strong electromagnetic field, we may neglect the pressure gradient in the momentum equation [5], and then, taking into account the viscous friction of the gas, the momemtum equation may be written approximately in the form

$$m\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2} + f, \qquad u(0) = 0, \qquad u(2h) = 0$$

$$(m = G/2h = \text{const}).$$

Here μ is the dynamical viscosity coefficient, x is the coordinate measured along the axis of the channel, G is the mass rate of gas flow per second, and f is the density of the accelerating electromagnetic force. If we neglect the initial gas momentum (u = 0 for x = 0) and take f = const, then, making use of the solution found above, we find that the ratio of the frictional force integrated along x to the electromagnetic force, also integrated along x, is equal to $\eta(\tau)$ for $\vartheta_i = 0$, $a = 0$, $\tau = \tau_v = \mu x / h^2 m$, where the function $\eta(\tau)$ is determined from (12) and (18).

We shall consider the energy equation. For a strong electromagnetic field, and on condition that the Joule dissipation in the channel (j^2/σ is equal in order of magnitude to the work done by the electromagnetic force in unit time (uf), we may neglect the product of the velocity and the pressure gradient ($u\partial p/\partial x$) in the energy equation. We then have approximately

$$mc_p\frac{\partial T}{\partial x} = -\frac{\partial q_y}{\partial y} + w, \qquad T(x, 0) = T_w,$$

$$T(x, 2h) = T_w, \qquad T(0, y) = T_i.$$

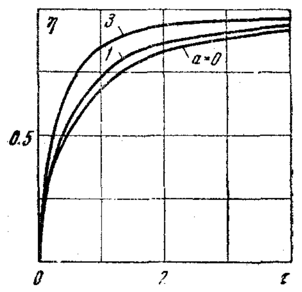

Fig. 5

If the heat flux is of the form (1), then we may use the solution already obtained for making estimates. The ratio of the amount of heat conducted per second through a length x and unit width of the channel wall to the Joule energy dissipated in a volume x × × 2h × 1 is equal to $\eta(\tau)$ where $\tau = \tau_q = \mu x / Ph^2 m$ (P is Prandtl's number). In the case of a fully ionized gas Prandtl's number is small $\tau_v \ll \tau_q$, and the effects associated with heat transfer are much stronger than the effects associated with the influence of viscosity. If the heat flux vector has a term proportional to the electric current density vector, then heat transfer proceeds with greater intensity, as shown above.

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28 May 1965 Moscow